

EFFECT ALGEBRAS ARE THE EILENBERG-MOORE CATEGORY FOR THE KALMBACH MONAD

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ABSTRACT. The Kalmbach monad is the monad that arises from the free-forgetful adjunction between bounded posets and orthomodular posets. We prove that the category of effect algebras is isomorphic to the Eilenberg-Moore category for the Kalmbach monad.

1. INTRODUCTION

In [6], Kalmbach proved the following theorem.

Theorem 1. *Every bounded lattice L can be embedded into an orthomodular lattice $K(L)$.*

The proof of the theorem is constructive, $K(L)$ is known under the name *Kalmbach extension* or *Kalmbach embedding*. In [10], Mayet and Navara proved that Theorem 1 can be generalized: every bounded poset P can be embedded in an orthomodular poset $K(P)$. In fact, as proved by Harding in [5], this K is then left adjoint to the forgetful functor from orthomodular posets to bounded posets. This adjunction gives rise to a monad on the category of bounded posets, which we call the *Kalmbach monad*.

For every monad (T, η, μ) on a category \mathbf{C} , there is a standard notion *Eilenberg-Moore category* \mathbf{C}^T (sometimes called the *category of algebras* or the *category of modules* for T). The category \mathbf{C}^T comes equipped with a canonical adjunction between \mathbf{C} and \mathbf{C}^T and this adjunction gives rise to the original monad T on \mathbf{C} .

In the present paper we prove that the Eilenberg-Moore category for the Kalmbach monad is isomorphic to the category of effect algebras.

2. PRELIMINARIES

We assume familiarity with basics of category theory, see [9, 1] for reference.

2.1. Bounded posets. A *bounded poset* is a structure $(P, \leq, 0, 1)$ such that \leq is a partial order on P , $0, 1 \in P$ are the bottom and top elements of (P, \leq) , respectively.

Let P_1, P_2 be bounded posets. A map $\phi : P_1 \rightarrow P_2$ is a *morphism of bounded posets* if and only if it satisfies the following conditions.

- $\phi(1) = 1$ and $\phi(0) = 0$.
- ϕ is isotone.

The category of bounded posets is denoted by **BPos**.

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2.2. Effect algebras. An *effect algebra* is a partial algebra $(E; \oplus, 0, 1)$ with a binary partial operation \oplus and two nullary operations $0, 1$ satisfying the following conditions.

- (E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.
- (E2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
- (E3) For every $a \in E$ there is a unique $a' \in E$ such that $a \oplus a'$ exists and $a \oplus a' = 1$.
- (E4) If $a \oplus 1$ is defined, then $a = 0$.

Effect algebras were introduced by Foulis and Bennett in their paper [3].

In an effect algebra E , we write $a \leq b$ if and only if there is $c \in E$ such that $a \oplus c = b$. It is easy to check that for every effect algebra E , \leq is a partial order on E . Moreover, it is possible to introduce a new partial operation \ominus ; $b \ominus a$ is defined if and only if $a \leq b$ and then $a \oplus (b \ominus a) = b$. It can be proved that, in an effect algebra, $a \oplus b$ is defined if and only if $a \leq b'$ if and only if $b \leq a'$. In an effect algebra, we write $a \perp b$ if and only if $a \oplus b$ exists.

Let E_1, E_2 be effect algebras. A map $\phi : E_1 \rightarrow E_2$ is called a *morphism of effect algebras* if and only if it satisfies the following conditions.

- $\phi(1) = 1$.
- If $a \perp b$, then $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$.

The category of effect algebras is denoted by **EA**. There is an evident forgetful functor $U : \mathbf{EA} \rightarrow \mathbf{BPos}$.

2.3. D-posets. In their paper [8], Chovanec and Kôpka introduced a structure called *D-poset*. Their definition is an abstract algebraic version the *D-poset of fuzzy sets*, introduced by Kôpka in the paper [7].

A D-poset is a system $(P; \leq, \ominus, 0, 1)$ consisting of a partially ordered set P bounded by 0 and 1 with a partial binary operation \ominus satisfying the following conditions.

- (D1) $b \ominus a$ is defined if and only if $a \leq b$.
- (D2) If $a \leq b$, then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$.
- (D3) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

Let D_1, D_2 be D-posets. A map $\phi : D_1 \rightarrow D_2$ is called a *morphism of D-posets* if and only if it satisfies the following conditions.

- $\phi(1) = 1$.
- If $a \leq b$, then $\phi(a) \leq \phi(b)$ and $\phi(b \ominus a) = \phi(b) \ominus \phi(a)$.

The category of D-posets is denoted by **DP**.

There is a natural, one-to-one correspondence between D-posets and effect algebras. Every effect algebra satisfies the conditions (D1)-(D3). When given a D-poset $(P; \leq, \ominus, 0, 1)$, one can construct an effect algebra $(P; \oplus, 0, 1)$: the domain of \oplus is given by the rule $a \perp b$ if and only if $a \leq 1 \ominus b$ and we then have $a \oplus b = 1 \ominus ((1 \ominus a) \ominus b)$. The resulting structure is then an effect algebra with the same \ominus as the original D-poset. It is easy to see that this correspondence is, in fact, an isomorphism of categories **DP** and **EA**.

Another equivalent structure was introduced by Giuntini and Greuling in [4]. We refer to [2] for more information on effect algebras and related topics.

The following lemma collects some well-known properties connecting the \oplus , \ominus and $'$ operations in effect algebras (or D-posets). Complete proofs can be found, for example, in Chapter 1 of [2]. We shall use these facts without an explicit reference.

Lemma 2.

(a) $a \leq b'$ iff $b \leq a'$ iff $a \oplus b$ exists and then

$$(a \oplus b)' = a' \ominus b = b' \ominus a.$$

(b) $a \leq b$ iff $a \oplus b'$ exists and then

$$(b \ominus a)' = a \oplus b'$$

(c) $a \leq c \ominus b$ iff $b \leq c \ominus a$ iff $a \oplus b \leq c$ and then

$$c \ominus (a \oplus b) = (c \ominus a) \ominus b = (c \ominus b) \ominus a$$

(d) $a \leq b \leq c$ iff $a \oplus (c \ominus b)$ exists and then

$$c \ominus (b \ominus a) = a \oplus (c \ominus b).$$

2.4. Orthomodular posets. An *orthomodular poset* is a structure $(A, \leq, ', 0, 1)$ such that $(A, \leq, 0, 1)$ is a bounded poset and $'$ is a unary operation (called *orthocomplementation*) satisfying the following conditions.

- $x \leq y$ implies $y' \leq x'$.
- $x'' = x$.
- $x \wedge x' = 0$.
- If $x \leq y'$, then $x \vee y$ exists.
- If $x \leq y$, then $x \vee (x \vee y')' = y$.

If $x \leq y'$, we say that x, y are *orthogonal*.

Let A_1, A_2 be orthomodular posets. A map $\phi : A_1 \rightarrow A_2$ is called a *morphism of orthomodular posets* if and only if it satisfies the following conditions.

- $\phi(1) = 1$.
- If $a \leq b'$, then $\phi(a) \leq \phi(b)'$ and $\phi(a \vee b) = \phi(a) \vee \phi(b)$.

Alternatively, we may define a morphism of orthomodular posets as order preserving, preserving the orthocomplementation, and preserving joins of orthogonal elements.

The category of orthomodular posets is denoted by **OMP**. An *orthomodular lattice* is an orthomodular poset that is a lattice. We remark that the usual category of orthomodular lattices, with morphisms preserving joins and meets is not a full subcategory of **OMP**.

If A is an orthomodular poset, then we may introduce a partial \oplus operation on A by the following rule: $x \oplus y$ exists iff $x \leq y'$ and then $x \oplus y := x \vee y$. The resulting structure is then an effect algebra. This gives us the object part of an evident full and faithful functor **OMP** \rightarrow **EA**.

2.5. Kalmbach construction. If $C = \{x_1, \dots, x_n\}$ is a finite chain in a poset P , we write $C = [x_1 < \dots < x_n]$ to indicate the partial order.

Definition 3. [6] Let P be a bounded poset, write

$$K(P) = \{C : C \text{ is a finite chain in } P \text{ with even number of elements}\}$$

Define a partial order on $K(P)$ by the following rule:

$$[x_1 < x_2 < \dots < x_{2n-1} < x_{2n}] \leq [y_1 < y_2 < \dots < y_{2k-1} < y_{2k}]$$

if for every $1 \leq i \leq n$ there is $1 \leq j \leq k$ such that

$$y_{2j-1} \leq x_{2i-1} < x_{2i} \leq y_{2j}.$$

Define a unary operation $C \mapsto C^\perp$ on $K(P)$ to be the symmetric difference with the set $\{0, 1\}$.

Originally, Kalmbach considered the construction only for lattices. If P is a bounded lattice, then $K(P)$ is a lattice as well. Moreover, $(K(P), \wedge, \vee, ', 0, 1)$ is an orthomodular lattice. However, as observed by Harding in [5], K is not an object part of a functor from the category of bounded lattices to the category of orthomodular lattices.

On the positive side, for any bounded poset P , $K(P)$ is an orthomodular poset (see [10]) and K can be made to a functor $K : \mathbf{BPos} \rightarrow \mathbf{OMP}$. Indeed, let $f : P \rightarrow Q$ be a morphism in \mathbf{BPos} and define $K(f) : K(P) \rightarrow K(Q)$ by the following rule.

For an arrow $f : P \rightarrow Q$ in \mathbf{BPos} , write

$$K(f)([x_1 < x_2 < \cdots < x_{2n-1} < x_{2n}]) = \{y \in Q : \text{card}(\{1 \leq i \leq n : f(x_i) = y\}) \text{ is odd}\}.$$

A more elegant way how to write the same rule is

$$K(f)([x_1 < x_2 < \cdots < x_{2n-1} < x_{2n}]) = \Delta_{i=1}^{2n} \{f(x_i)\},$$

where Δ is the symmetric difference of sets.

Then K is a functor. Moreover, as proved by Harding in [5], K is left-adjoint to the forgetful functor $U : \mathbf{OMP} \rightarrow \mathbf{BPos}$. Since every functor has (up to isomorphism) at most one adjoint, this can be viewed as an alternative definition of the Kalmbach construction.

The unit of the $K \dashv U$ adjunction is the natural transformation $\eta : \text{id}_{\mathbf{BPos}} \rightarrow UK$, given by the rule

$$\eta_P(a) = \begin{cases} [0 < a] & a > 0 \\ \emptyset & a = 0 \end{cases}$$

and the counit of the adjunction is the natural transformation $\epsilon : KU \rightarrow \text{id}_{\mathbf{OMP}}$ given by the rule

$$\epsilon_L([x_1 < \cdots < x_{2n}]) = (x_1^\perp \wedge x_2) \vee \cdots \vee (x_{2n-1}^\perp \wedge x_{2n}).$$

3. KALMBACH MONAD

Let \mathbf{C} be a category. A *monad* on \mathbf{C} can be defined as a monoid in the strict monoidal category of endofunctors of \mathbf{C} . Explicitly, a monad on \mathbf{C} is a triple (T, η, μ) , where $T : \mathbf{C} \rightarrow \mathbf{C}$ is an endofunctor of \mathbf{C} and η, μ are natural transformations $\eta : \text{id}_{\mathbf{C}} \rightarrow T$, $\mu : T^2 \rightarrow T$ satisfying the equations $\mu \circ T\mu = \mu \circ \mu T$ and $\mu \circ T\eta = \mu \circ \eta T = 1_T$.

Every adjoint pair of functors $F : \mathbf{C} \rightleftarrows \mathbf{D} : G$, with F being left adjoint, gives rise to a monad $(FG, \eta, G\epsilon F)$ on \mathbf{C} .

Let (T, η, μ) be a monad on a category \mathbf{C} . Recall, that the *Eilenberg-Moore category* for (T, η, μ) is a category (denoted by \mathbf{C}^T), such that objects (called *algebras for that monad*) of \mathbf{C}^T are pairs (A, α) , where $\alpha : T(A) \rightarrow A$, such that the

diagrams

$$(1) \quad \begin{array}{ccc} A & \xrightarrow{\eta_A} & T(A) \\ & \searrow 1_A & \downarrow \alpha \\ & & A \end{array}$$

$$(2) \quad \begin{array}{ccc} T^2(A) & \xrightarrow{T(\alpha)} & T(A) \\ \mu_A \downarrow & & \downarrow \alpha \\ T(A) & \xrightarrow{\alpha} & A \end{array}$$

commute. A morphism of algebras $h : (A_1, \alpha_1) \rightarrow (A_2, \alpha_2)$ is a \mathbf{C} -morphism such that the diagram

$$\begin{array}{ccc} T(A) & \xrightarrow{T(h)} & T(B) \\ \alpha_1 \downarrow & & \downarrow \alpha_2 \\ A & \xrightarrow{h} & B \end{array}$$

commutes.

Consider now the adjunction $K \dashv U$ between the categories, \mathbf{BPos} and \mathbf{OMP} from the preceding section. This adjunction gives rise to a monad on \mathbf{BPos} , which we will denote (T, η, μ) . Explicitly, $T = U \circ K$, η remains the same and $\mu = U\epsilon K$ turns out to be given by the following rule

$$\mu_P([C_1 < C_2 < \dots < C_{2n}]) = \Delta_{i=1}^{2n} C_i,$$

where $[C_1 < C_2 < \dots < C_{2n}]$ is a chain of even length of chains of even length, that means, an element of $T^2(P)$, and Δ is the symmetric difference of sets.

Theorem 4. *The category of effect algebras is isomorphic to the Eilenberg-Moore category for the Kalmbach monad.*

Proof. From now on, let U be the forgetful functor $U : \mathbf{EA} \rightarrow \mathbf{BPos}$. Let us define a functor $G : \mathbf{EA} \rightarrow \mathbf{BPos}^T$. For an effect algebra A , define $m_A : T(U(A)) \rightarrow U(A)$ by the rule

$$m_A([x_1 < x_2 < \dots < x_{2n-1} < x_{2n}]) = (x_2 \ominus x_1) \oplus \dots \oplus (x_{2n} \ominus x_{2n-1}).$$

We claim that $G(A) = (U(A), m_A)$ is an algebra for the Kalmbach monad. We need to prove that the diagrams (1) and (2) commute. Clearly, for every $x \in U(A)$,

$$(m_A \circ \eta_{U(A)})(x) = m_A([0 < x]) = (x \ominus 0) = x,$$

and we see that the triangle diagram (1) commutes. Consider now the square diagram (2): the elements of $T^2(U(A))$ are chains of chains of elements of $U(A)$; let $[C_1 < C_2 < \dots < C_{2n}] \in T^2(U(A))$. Note that $C_i < C_j$ implies that $m_A(C_i) < m_A(C_j)$, so the elements of the sequence $(m_A(C_1), \dots, m_A(C_{2n}))$ are pairwise distinct. Therefore,

$$\begin{aligned} (m_A \circ T(m_A))([C_1 < C_2 < \dots < C_{2n}]) &= \\ m_A([m_A(C_1) < m_A(C_2) < \dots < m_A(C_{2n})]) &= \\ (m_A(C_2) \ominus m_A(C_1)) \oplus \dots \oplus (m_A(C_{2n}) \ominus m_A(C_{2n-1})) & \end{aligned}$$

Note that, if $C < D$ in $T(U(A))$, then $C\Delta D < D$, $m_A(C) < m_A(D)$ and $m_A(D) \ominus m_A(C) = m_A(C\Delta D)$. Using these facts,

$$\begin{aligned}
(m_A \circ \mu_{U(A)})([C_1 < C_2 < \dots < C_{2n}]) &= m_A(C_1\Delta C_2\Delta \dots \Delta C_{2n}) = \\
&= m_A((C_1\Delta C_2\Delta \dots \Delta C_{2n-1})\Delta C_{2n}) = \\
&= m_A(C_{2n}) \ominus m_A(C_1\Delta C_2\Delta \dots \Delta C_{2n-1}) = \\
m_A(C_{2n}) \ominus (m_A(C_{2n-1}) \ominus m_A(C_1\Delta C_2\Delta \dots \Delta C_{2n-2})) &= \\
(m_A(C_{2n}) \ominus m_A(C_{2n-1})) \oplus m_A(C_1\Delta C_2\Delta \dots \Delta C_{2n-2}).
\end{aligned}$$

The desired equality now follows by a simple induction.

If $f : A \rightarrow B$ is a morphism of effect algebras, we define $G(f) = U(f)$. We need to prove that the diagram

$$\begin{array}{ccc}
TU(A) & \xrightarrow{TU(f)} & TU(A) \\
m_A \downarrow & & \downarrow m_B \\
U(A) & \xrightarrow{U(f)} & U(B)
\end{array}$$

commutes. After some simple steps, this reduces to the following equality in B :

$$(3) \quad (f(x_2) \ominus f(x_1)) \oplus \dots \oplus (f(x_{2n}) \ominus f(x_{2n-1})) = m_B(\Delta_{i=1}^n \{f(x_i)\}),$$

for each $[x_1 < \dots < x_{2n}] \in T(U(A))$.

Let us define an auxiliary function $k : T(U(A)) \rightarrow \mathbb{N}$: for $C = [x_1 < x_2 < \dots < x_{2n}] \in T(U(A))$, $k(C)$ is the number of equal consecutive pairs in the sequence $(f(x_1), \dots, f(x_n))$, that means, $k(C)$ is the cardinality of the set $\{i : f(x_i) = f(x_{i+1})\}$.

To prove the equality (3), we use induction with respect to $k(C)$. If $k(C) = 0$, then the equality (3) clearly holds.

If $k(C) > 0$, then let us pick some i with $f(x_i) = f(x_{i+1})$. Then $\{f(x_i)\} \Delta \{f(x_{i+1})\} = \emptyset$ and we may skip them on the right hand side of (3).

If i is odd, then $f(x_{i+1}) \ominus f(x_i) = 0$ and we may delete that term from the left-hand side of (3). If i is even, then

$$(f(x_i) \ominus f(x_{i-1})) \oplus (f(x_{i+2}) \ominus f(x_{i+1})) = f(x_{i+2}) \ominus f(x_{i-1})$$

and we may simplify the left-hand side of (3) accordingly.

So (3) is true if and only if it is true for the chain $C - \{x_i, x_{i+1}\}$. Clearly, $k(C - \{x_i, x_{i+1}\}) = k(C) - 1$ and we have completed the induction step.

Let (A, α) be an algebra for the Kalmbach monad. Let us define a partial operation \ominus on the bounded poset A given by this rule: $b \ominus a$ is defined if and only if $a \leq b$ and

$$b \ominus a = \begin{cases} 0 & a = b \\ \alpha([a < b]) & a < b \end{cases}$$

We claim that $E(A, \alpha) = (A, \leq, \ominus, 0, 1)$ is then a D-poset, hence an effect algebra.

The axiom (D1) follows by definition.

Before we prove the other two axioms, let us note that for all $a \in A$, $a \ominus 0 = a$. Indeed, if $0 < a$ then the triangle diagram (1) implies that $a = \alpha([0 < a]) = a \ominus 0$ and for $a = 0$ we obtain $a \ominus 0 = 0 \ominus 0 = 0$ by definition of \ominus .

To prove (D2), let $a, b \in A$ be such that $a \leq b$.

Let us prove that $b \ominus a \leq b$. If $a < b$, then $[a < b] \leq [0 < b]$ in the poset $T(A)$ and

$$b \ominus a = \alpha([a < b]) \leq \alpha([0 < b]) = b \ominus 0 = b.$$

If $a = b$ then $b \ominus a = 0 \leq b$.

Let us prove that $b \ominus (b \ominus a) = a$. There are three possible cases.

- (D2.1) Suppose that $0 < a < b$. Then, $[a < b] < [0 < b]$ in $T(A)$ and hence $[a < b] < [0 < b] \in T^2(A)$. Suppose that $\alpha([a < b]) = \alpha([0 < b])$. From the commutativity of the square (2) we obtain

$$\begin{array}{ccc} [a < b] < [0 < b] & \xrightarrow{T(\alpha)} & \emptyset \\ \mu_A \downarrow & & \downarrow \alpha \\ [0 < a] & \xrightarrow{\alpha} & \alpha([0 < a]) = 0 \end{array}$$

However, $0 < a = \alpha([0 < a]) = 0$ is false and we have proved that $\alpha([a < b]) < \alpha([0 < b])$. Chasing the element $[a < b] < [0 < b]$ around the square

$$\begin{array}{ccc} [a < b] < [0 < b] & \xrightarrow{T(\alpha)} & [\alpha([a < b]) < \alpha([0 < b])] \\ \mu_A \downarrow & & \downarrow \alpha \\ [0 < a] & \xrightarrow{\alpha} & \alpha([0 < a]) = \alpha([\alpha([a < b]) < \alpha([0 < b])]) \end{array}$$

gives us the equality in the bottom right corner, meaning that $b \ominus (b \ominus a) = a$.

- (D2.2) Suppose that $0 = a$. We already know that $b \ominus 0 = b$ and we may compute

$$b \ominus (b \ominus a) = b \ominus (b \ominus 0) = b \ominus b = 0 = a.$$

- (D2.3) Suppose that $a = b$. Then

$$b \ominus (b \ominus a) = b \ominus 0 = b = a.$$

To prove (D3), let $a, b, c \in A$ be such that $a \leq b \leq c$.

Let us prove that $c \ominus b \leq c \ominus a$. If $a = b$, there is nothing to prove. If $b = c$, then $b \ominus c = 0 \leq c \ominus a$. Assume that $a < b < c$. Then $[b < c] < [a < c]$ and

$$c \ominus b = \alpha([b < c]) \leq \alpha([a < c]) = c \ominus a.$$

Let us prove that $b \ominus a = (c \ominus a) \ominus (c \ominus b)$.

- (D3.1) Suppose that $a < b < c$ and assume that $\alpha([b < c]) = \alpha([a < c])$. The square

$$\begin{array}{ccc} [b < c] < [a < c] & \xrightarrow{T(\alpha)} & \emptyset \\ \mu_A \downarrow & & \downarrow \alpha \\ [a < b] & \xrightarrow{\alpha} & \alpha([a < b]) = 0 \end{array}$$

gives us $\alpha([a < b]) = 0$, so $b \ominus a = 0$. However, using only the properties of \ominus we already proved,

$$b = b \ominus 0 = b \ominus (b \ominus a) = a < b,$$

which is false. Thus, assuming $\alpha([b < c]) < \alpha([a < c])$ the square

$$\begin{array}{ccc} [b < c] < [a < c] & \xrightarrow{T(\alpha)} & [\alpha([b < c]) < \alpha([a < c])] \\ \mu_A \downarrow & & \downarrow \alpha \\ [a < b] & \xrightarrow{\alpha} & \alpha([a < b]) = \alpha([\alpha([b < c]) < \alpha([a < c])]) \end{array}$$

gives us the equality in the bottom right corner meaning that

$$b \ominus a = (c \ominus a) \ominus (c \ominus b).$$

(D3.2) Suppose that $b = c$. Then

$$(c \ominus a) \ominus (c \ominus b) = (c \ominus a) \ominus 0 = c \ominus a = b \ominus a.$$

(D3.3) If $a = b$, then there is nothing to prove.

If is now easy to check that an arrow $h : (A, \alpha) \rightarrow (B, \beta)$ in \mathbf{BPos}^T is, at the same time, a morphism of D-posets (and thus, a morphism of effect algebras) $E(A, \alpha) \rightarrow E(B, \beta)$. Indeed, $h(0) = 0$ and $h(1) = 1$. If $a < b$ and $h(a) < h(b)$, then

$$h(b \ominus a) = h(\alpha([a < b])) = \beta(T(h)([a < b])) = \beta([h(a) < h(b)]) = h(b) \ominus h(a).$$

If $a < b$ and $h(a) = h(b)$ then

$$h(b \ominus a) = h(\alpha[a < b]) = \beta(T(h)([a < b])) = \beta(\emptyset) = 0 = h(b) \ominus h(a).$$

Therefore E is a functor from \mathbf{BPos}^T to \mathbf{EA} .

It remains to prove that E, G are mutually inverse functors. Let A be an effect algebra. We claim that $EG(A) = A$. The underlying poset of $EG(A)$ and A is the same. For all $a < b$,

$$b \ominus_{EG(A)} a = m_A([a < b]) = b \ominus_A a,$$

hence $EG(A) = A$. It is obvious that for every morphism $f : A \rightarrow B$ of effect algebras $EG(f) = f$, since both E and G preserve the underlying poset maps.

Let (A, α) be an algebra for the Kalmbach monad. We claim that $(A, \alpha) = GE(A, \alpha)$, that means, for all $[x_1 < \dots < x_{2k}] \in T(A)$,

$$(4) \quad \alpha([x_1 < x_2 < \dots < x_{2k-1} < x_{2k}]) = (x_2 \ominus x_1) \oplus \dots \oplus (x_{2k} \ominus x_{2k-1}),$$

where the \oplus, \ominus on the right-hand side are taken in $E(A, \alpha)$.

To prove (4) we need an auxiliary claim: for every $C \in T(A)$ and an upper bound u of C with $u \notin C$, $\alpha(C \Delta [0 < u]) = \alpha([\alpha(C) < u])$. This is easily seen by chasing the element $[C < [0 < u]] \in T^2(A)$ around the square (2).

Clearly, equality (4) is true for $k = 0$. Suppose it is valid for some $k = n \in \mathbb{N}$. Then, for $k = n + 1$, equality (4) is then equivalent to

$$\begin{aligned} \alpha([x_1 < x_2 < \dots < x_{2n-1} < x_{2n} < x_{2n+1} < x_{2n+2}]) = \\ \alpha([x_1 < x_2 < \dots < x_{2n-1} < x_{2n}]) \oplus (x_{2n+2} \ominus x_{2n+1}) \end{aligned}$$

Put $C = [x_1 < x_2 < \cdots < x_{2n-1} < x_{2n}]$. Using the definition of \ominus in $E(A, \alpha)$ and applying the auxiliary claim twice we obtain

$$\begin{aligned}
& \alpha(C) \oplus (x_{2n+2} \ominus x_{2n+1}) = \\
& x_{2n+2} \ominus (x_{2n+1} \ominus \alpha(C)) = \\
& x_{2n+2} \ominus \left(\alpha([\alpha(C) < x_{2n+1}]) \right) = \\
& x_{2n+2} \ominus \left(\alpha(C \Delta [0 < x_{2n+1}]) \right) = \\
& \alpha([\alpha(C \Delta [0 < x_{2n+1}]) < x_{2n+2}]) = \\
& \alpha([C \Delta [0 < x_{2n+1}] \Delta [0 < x_{2n+2}]]) = \\
& \alpha([C \Delta [x_{2n+1} < x_{2n+2}]]) = \\
& \alpha([x_1 < x_2 < \cdots < x_{2n-1} < x_{2n} < x_{2n+1} < x_{2n+2}])
\end{aligned}$$

It is obvious that for every morphism f in \mathbf{BPos}^T , $GE(f) = f$. \square

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